Relativistic Equilibrium Distribution by Relative Entropy Maximization

Tadas K. Nakamura

Fukui Prefectural University, 910-1195 Fukui, JAPAN

Abstract. - The equilibrium state of a relativistic gas has been calculated based on the maximum entropy principle. Though the relativistic equilibrium state was long believed to be the Jüttner distribution, a number of papers have been published in recent years proposing alternative equilibrium states. However, some of these papers do not pay enough attention to the covariance of distribution functions, resulting confusion in equilibrium states. Starting from a fully covariant expression to avoid this confusion, it has been shown in the present paper that the Jüttner distribution is the maximum entropy state if we assume the Lorentz symmetry.

Introduction. – Little after the establishment of the theory of relativity, the equilibrium particle distribution of a relativistic gas was investigated. The distribution obtained, which is called Jüttner distribution [1,2], has been long and widely believed. However, relatively recent years a number of papers have been published proposing equilibrium distribution functions other than the Jüttner distribution ([3–7] and references therein). Dunkel and coworkers [7,8] have examined the discrepancy in the equilibrium distributions as the maximum entropy state, and showed that the difference comes from the choice of the reference measure.

The maximum entropy state cannot be uniquely determined when one naively defines the entropy such as $S = -\int f(\mathbf{x}, \mathbf{v}) \ln f(\mathbf{x}, \mathbf{v}) \, d\mathbf{x} d\mathbf{v}$ (symbols have conventional meaning in the present paper unless otherwise stated). For instance, the result would be different if we rewrite distribution function as a function of momentum $\mathbf{p}$ instead of velocity $\mathbf{v}$. To overcome this difficulty, it was proposed in Ref [7] to maximize the following relative entropy

$$S = -\int f(\mathbf{x}, \mathbf{v}) \ln f(\mathbf{x}, \mathbf{v}) / \rho(\mathbf{x}, \mathbf{v}) \, d\mathbf{x} d\mathbf{v} \,, \tag{1}$$

based on a given reference measure ρ. In the above expression f is the phase space distribution of particles and ρ is the reference measure [7]. In this paper we denote a three vector by a bold font (e.g., $\mathbf{x}$) and a four vector by an upper bar (e.g., $\bar{x}$). Each component of a vector is represented by a subscript or a superscript (e.g., x_μ or x^μ).

The equilibrium distribution is uniquely determined by maximizing the relative entropy once the reference measure is given. The mathematical procedure in this approach is essentially the same as the one utilized in Ref [2] to derive the Jüttner distribution. What is called "a priori probability" in Ref [2] plays the same role as the reference measure in

Ref [7].

Two possibilities for the reference frame were suggested in Ref [7]. One is the constant distributions a function of momentum, and the Jïtner distribution is obtained from this measure. This calculation is essentially the same as the one in Ref [2]. Another possibility suggested in Ref [7] which is inversely proportional to the energy. It was argued this measure is derived from the Lorentz symmetry in Ref [7] and the result is the alternative equilibrium distribution proposed in recent papers. However, as we will see in the present paper, there is a confusion on the relativistic phase space density in this argument. The Lorentz invariant reference measure is the same as the one in Ref [2], i.e., the constant measure, which gives the Jüttner distribution.

There is a misleading point in defining a phase space density such as a particle distribution in relativity. When we express a phase space density as the time evolution of the density in a six (three space + three momentum) dimensional phase space, it appears to be a Lorentz invariant. Actually, it can be proved [9] (see also [2,10]) that $f(t, \mathbf{x}, \mathbf{p}) = f(t', \mathbf{x}', \mathbf{p}')$ when the two sets of coordinates $(t, \mathbf{x}, \mathbf{p})$ and $(t', \mathbf{x}', \mathbf{p}')$ are related by the Lorentz transform, in other words, they are the same point in the spacetime denoted by different reference coordinates. However, this does not mean $f(t, \mathbf{x}, \mathbf{p})d\mathbf{x}d\mathbf{p} = f(t', \mathbf{x}', \mathbf{p}')d\mathbf{x}'d\mathbf{p}'$ because $\mathbf{x}$ and $\mathbf{x}'$ do not belong to the same spatial volume. In this sense, phase a space density in the form of $f(t, \mathbf{x}, \mathbf{p})$ is not covariant but frame dependent. It seems that some of recent papers do not pay enough attention to this fact, resulting confision in treating Lorentz transfrom.

In the present paper, we examine this confusing point by starting from the fully covariant distribution function proposed by Hakim [11], and the result shows the reference measure should be constant to satisfy the full Lorentz symmetry; the one introduced in Ref [7] is invariant under the Lorentz transform only in the momentum space. This result means the maximum entropy state with Lorentz symmetry must be the Jüttner distribution.

Relativistic Phase Space Density. –   Let us suppose a relativistic gas as an example. The conservation law of its particle number is expressed in the form of flux divergence in relativity:

$$\frac{\partial}{\partial x_\mu} J_\mu = 0 \, , \tag{2}$$

where

$$J_\mu = n_0 u_\mu \, . \tag{3}$$

is the four flux derived from the proper number density n_0 and the four velocity of the matter u_μ. When we split the spacetime as $t_\Sigma = x_{\Sigma 0}$ and $\mathbf{x}_\Sigma = (x_{\Sigma 1}, x_{\Sigma 2}, x_{\Sigma 3})$ by choosing a specific reference frame Σ, the above conservation is written as

$$\frac{\partial}{\partial t} n_\Sigma(t_\Sigma, \mathbf{x}_\Sigma) + \nabla \mathbf{J}_\Sigma(t_\Sigma, \mathbf{x}_\Sigma) = 0 \, . \tag{4}$$

In the above expression, $n_\Sigma = J_{\Sigma 0}$ and $\mathbf{J}_\Sigma = (J_{\Sigma 1}, J_{\Sigma 2}, J_{\Sigma 3})$ are the number density and flux in the three dimensional space; the subscript Σ is to explicitly express the frame dependence.

When we decompose the spacetime in another reference frame Σ', obviously $n_{\Sigma'}$ is different from n_Σ. Moreover, n_Σ and $n_{\Sigma'}$ cannot be related with a Jacobian $\partial \mathbf{x}_\Sigma / \partial \mathbf{x}_{\Sigma'}$ as

$$n_\Sigma d\mathbf{x}_\Sigma = n_{\Sigma'} \frac{\partial \mathbf{x}_\Sigma}{\partial \mathbf{x}_{\Sigma'}} d\mathbf{x}_{\Sigma'} \, , \tag{5}$$

because $\mathbf{x}_\Sigma$ and $\mathbf{x}_\Sigma$ belong to different spacelike volumes. There is no function to relate $\mathbf{x}_\Sigma$ and $\mathbf{x}_{\Sigma'}$ as $\mathbf{x}_\Sigma = \mathbf{X}(\mathbf{x}_{\Sigma'})$ where $\mathbf{X}$ that does not depend on the time coordinate.

The above argument on the number density in a three dimensional space is also valid for phase space densities in a six dimensional space. A phase space density is often expressed

as $f(t, \mathbf{x}, \mathbf{p})$ and it should be denoted in our notation as $f_\Sigma(t_\Sigma, \mathbf{x}_\Sigma, \mathbf{p}_\Sigma)$ because the expression is based on a specific choice of the reference frame like n_Σ in (4). However, it is generally believed that the phase space density is unchanged under the Lorentz transform. This is true in the sense that the value of the phase space density is unchanged [2, 9, 10], but $f_\Sigma(t_\Sigma, \mathbf{x}_\Sigma, \mathbf{p}_\Sigma)$ is defined only on a space volume in a specific reference frame, and not directly applicable to other reference frames.

To correctly treat the phase space density, we derive the frame-dependent phase space density $f_\Sigma(t_\Sigma, \mathbf{x}_\Sigma, \mathbf{p}_\Sigma)$ from the fully covariant expression proposed by Hakim [11]. The relativistic particle distribution $N(\bar{x}, \bar{p})$ is defined such that $\bar{j}$ in the following expression becomes the particle four-current:

$$j_\mu(\bar{x}) = \int d_4 p \, 2m u_\mu N(\bar{x}, \bar{p}) \, \theta(p^0) \delta(p^\mu p_\mu - m^2) \,, \tag{6}$$

where θ and δ are the theta and delta functions, and m is the particle rest mass.

In the above expression, $N(\bar{x}, \bar{p})$ can be interpreted as the proper density of the fluid element that has the four velocity $\bar{u} = \bar{p}/m$, just like n_0 in (3). Thus its covariant form must be a four vector, which is expressed as $N(\bar{x}, \bar{p})\bar{u}$, like $\bar{J}$ in (3). The delta function is due to the energy shell and the theta function is to discard the negative energy solution. Hakim [11] has introduced the above expression for the distribution of particle number, however, it is generally valid for a conserved density flowing with the four velocity $\bar{u}$, therefore, it can be applied to a probability distribution or a reference measure to calculate entropy in the following.

When we pick up one reference frame Σ and denote its unit vectors in each coordinate direction as $(\bar{e}_{\Sigma t}, \bar{e}_{\Sigma x}, \bar{e}_{\Sigma y}, \bar{e}_{\Sigma z})$, an arbitrary point in the eight dimensional phase space $(\bar{x}, \bar{p})$ can be represented in this reference frame as

$$t_\Sigma = e_{\Sigma t}^\mu x_\mu, \quad \mathbf{x}_\Sigma = (e_{\Sigma x}^\mu x_\mu, e_{\Sigma y}^\mu x_\mu, e_{\Sigma z}^\mu x_\mu) \,, \tag{7}$$

and

$$E_\Sigma = e_{\Sigma t}^\mu p_\mu, \quad \mathbf{p}_\Sigma = (e_{\Sigma x}^\mu p_\mu, e_{\Sigma y}^\mu p_\mu, e_{\Sigma z}^\mu p_\mu) \,. \tag{8}$$

A frame-dependent phase space density $f_\Sigma(t_\Sigma, \mathbf{x}_\Sigma, \mathbf{p}_\Sigma)$ is then calculated from $N(\bar{x}, \bar{p})$ as

$$\begin{aligned} f_\Sigma(t_\Sigma, \mathbf{x}_\Sigma, \mathbf{p}_\Sigma) &= 2m \int e_{\Sigma t}^\mu u_\mu N(\bar{X}, \bar{P}) \, \theta(p^0) \delta(E_\Sigma^2 - \mathbf{p}_\Sigma^2 - m^2) \, dE_\Sigma \\ &= \frac{m e_{\Sigma t}^\mu u_\mu}{E_\Sigma} N(\bar{X}, \bar{P}) = N(\bar{X}, \bar{P}) \,, \end{aligned} \tag{9}$$

where $\bar{u} = \bar{p}/m$, and $\bar{X}$ and $\bar{P}$ are the covariant expression of the four dimensional position and momentum correspond to $(t, \mathbf{x}_\Sigma, \mathbf{p}_\Sigma)$,

$$\bar{X} = t_\Sigma \bar{e}_{\Sigma t} + x_\Sigma \bar{e}_{\Sigma x} + y_\Sigma \bar{e}_{\Sigma y} + z_\Sigma \bar{e}_{\Sigma z} \,, \tag{10}$$

and

$$\bar{P} = \sqrt{\mathbf{p}^2 + m^2} \, \bar{e}_{\Sigma t} + p_{\Sigma x} \bar{e}_{\Sigma x} + p_{\Sigma y} \bar{e}_{\Sigma y} + p_{\Sigma z} \bar{e}_{\Sigma z} \,. \tag{11}$$

From (9) van Kampen [9] concluded that f is unchanged under the Lorentz transform (his derivation is different from ours, but the result is the same). He considered the above result is purely kinematical. It is true in the sense that no equation of motion is required for (9), however, it implicitly includes kinetics in the expression of the energy shell. For example, if the relativistic kinetics were such that the energy shell is expressed as $4m^3 \delta(E_\Sigma^4 - \mathbf{p}_\Sigma^4 - m^4)$, (9) would be

$$f_\Sigma(t_\Sigma, \mathbf{x}_\Sigma \mathbf{p}_\Sigma) = \frac{m^3 e_{\Sigma t}^\mu u_\mu}{E_\Sigma^3} N(\bar{X}, \bar{P}) = \frac{m^2}{E_\Sigma^2} N(\bar{X}, \bar{P}) \,, \tag{12}$$

which means the value of f changes under the Lorentz transform. This example demonstrates the fact that f_Σ is not identical to N, but should be derived from N.

Lorentz Invariant Reference Frame. – In (9) we assumed the spatial coordinates $(t_\Sigma, \mathbf{x}_\Sigma)$ and the momentum coordinates $(E_\Sigma, \mathbf{p}_\Sigma)$ are defined in the same reference frame Σ. Mathematically the reference frames to define spatial and momentum coordinates do not have to be the same; we may have a phase space density whose spatial coordinates are defined in Σ and momentum coordinates are in Σ' as in the following form:

$$
\begin{aligned}
f_{\Sigma\Sigma'}(t_\Sigma, \mathbf{x}_\Sigma, \mathbf{p}_{\Sigma'}) &= 2m \int e_{\Sigma t}^\mu u_\mu N(\bar{X}, \bar{P}')\, \theta(p^0)\delta(E_{\Sigma'}^2 - \mathbf{p}_{\Sigma'}^2 - m^2)\, dE_{\Sigma'} \\
&= \frac{m e_{\Sigma t}^\mu u_\mu}{E_{\Sigma'}} N(\bar{X}, \bar{P}')\,,
\end{aligned} \tag{13}
$$

with

$$
\bar{P}' = E_{\Sigma'}\bar{e}_{\Sigma't} + p_{\Sigma'x}\bar{e}_{\Sigma'x} + p_{\Sigma'y}\bar{e}_{\Sigma'y} + p_{\Sigma'z}\bar{e}_{\Sigma'z}\,. \tag{14}
$$

From the above expression we understand that the factor of $e_{\Sigma t}^\mu u_\mu$ comes from the spatial Lorentz transform whereas the factor of $1/E_{\Sigma'}$ is due to the transform in the momentum space. They are canceled out when $\Sigma = \Sigma'$ and $f_{\Sigma\Sigma}$ becomes unchanged as seen in the previous section. This fact also indicates the phase space density is not a covariant expression; if it were covariant, $f_{\Sigma\Sigma'}$ should be unchanged even when $\Sigma \neq \Sigma'$.

Since $f_{\Sigma\Sigma}$ and $f_{\Sigma\Sigma'}$ are the densities defined on a same spatial volume in Σ, we can relate them by

$$
\frac{1}{E_\Sigma} f_{\Sigma\Sigma}\, d\mathbf{p}_{\Sigma'} d\mathbf{x}_\Sigma = \frac{1}{E_{\Sigma'}} f_{\Sigma\Sigma'}\, d\mathbf{p}_{\Sigma'} d\mathbf{x}_\Sigma\,. \tag{15}
$$

When we apply the above result to the reference measure ρ to calculate the relative entropy, it has the same meaning as Equation (34) in Ref [7]. If the measure ρ is to be invariant under the transform of $\rho_{\Sigma\Sigma} \to \rho_{\Sigma\Sigma'}$, it must be

$$
\rho(p_\Sigma) \propto \frac{1}{E_\Sigma}\,, \tag{16}
$$

which is suggested in Ref [7]. However, as seen from (15), the Lorentz transform in this context is in the momentum space only and the spatial volume to define the measure ρ is unchanged.

The present paper proposes that the measure should have the Lorentz symmetry under the transform both in space and momentum coordinates: $\rho_{\Sigma\Sigma} \to \rho_{\Sigma'\Sigma'}$. Then we have to choose the phase space density defined by (9) instead of (13) for the reference measure. As discussed above, two phase space densities with different reference frames Σ and Σ' is not directly connected with a equation such as (15). The Lorentz symmetry in this case means the mathematical expression is unchanged under the transform, and this is satisfied when $N(\bar{x}, \bar{p})$ is constant. Therefore we obtain

$$
\rho(p_\Sigma) = \text{constant}\,, \tag{17}
$$

in the reference frame Σ instead of (16). Following the relative entropy maximization procedure proposed in Ref [7] we obtain the Jüttner distribution as

$$
\phi(p_\Sigma) \propto \exp(-\beta E_\Sigma)\,. \tag{18}
$$

by maximizing the relative entropy in (1).

Concluding Remarks. – It has been shown in the present paper the maximum entropy state based on the Lorentz symmetry is the Jüttner distribution. Recent years a number of papers have been published claiming the relativistic equilibrium state is different from the long believed Jüttner distribution. Dunkel and coworkers [7,8] have shed a light to this controversy by pointing the importance of the reference measure in the maximum

entropy approach.. They have shown that the difference of the reference measure causes the difference of the equilibrium distribution as the maximum entropy state.

Two typical reference measures were suggested in Ref [7]. One is constant as a function of p and the other is inversely proportional to the energy. In Ref [7] it is conjectured the former is derived from the invariance of momentum transition, and the latter comes from the Lorentz symmetry. However, as we have seen in the present paper, the reference measure with Lorentz symmetry is also found to be constant when we correctly formulate the covariance of relativistic phase space density.

The constant reference measure we derived in this paper corresponds to the constant "prior probability" employed by Synge [2]. The information theory was developed long after the days of Synge, therefore, he did not know the modern concepts such as relative entropy or reference measure. Nevertheless, his calculation is quite similar to ours, and the result is the same Jüttner distribution. (The author guesses his basic idea historically comes from the probabilistic interpretation of the entropy by Boltzmann in his late years [12].)

Therefore, the argument in the present paper might seem just another interpretation of Synge's result with information theory if one believes his derivation. However, considerable number of papers have been published recently against the Jüttner distribution and it is important to clarify the foundation of the maximum entropy process based on information theory. Moreover, it has become clear in the present paper what causes the confusion of the reference measure (the difference of $\rho_{\Sigma'\Sigma}$ and $\rho_{\Sigma'\Sigma'}$).

The result in the present paper strongly suggests that the relativistic equilibrium state is the Jüttner distribution. There are papers in favor of the Jüttner distribution in the recent controversy. Debbasch [13] critically reviewed the theories proposing alternatives to the Jüttner distribution. He examined the relative entropy approach in Ref [7] and showed the result would be inconsistent unless the reference measure is constant.

Also there is a result of numerical experiment that supports the Jüttner distribution [14]. It was aregued in Ref [8] that the distribution measured in Ref [14] is based on what they call "coordinate-time", and and the modified distribution would be obtained if it is defined with "proper-time". In this sense, what we examined in the present paper is the one with "coordinate-time", in agreement with the numerical experiment.

It has been known, but has not been well recognized, that the a conserved quantity (energy-momentum, particle number etc.) distributed over a finite volume is not a Lorentz invariant quantity because it belongs to a different time slice of the volume's world tube. Confusions on this point have caused controversy on the relativistic thermodynamics ([15, 16] and references therein).

To treat this point correctly any spatial density must be expressed by a flux four vector as a covariant form. The density in the phase space is no exception. However, the phase space density in the form of $f(t, \mathbf{x}, \mathbf{p})$ is often regarded as a covariant expression since the value f is unchanged under the Lorentz transform.

As discussed in Section 2, the expression of $f(t, \mathbf{x}, \mathbf{p})$ is frame dependent since it is defined on a three dimensional space volume in a specific reference frame. It seems some of recent papers do not pay enough attention to this point, and treat the phase space density in a confusing way. In the present paper we start with the fully covariant expression of the phase space density [11] to avoid this confusion. We have seen that the reference measure with Lorentz symmetry is constant as a function of momentum. Consequently the maximum entropy state with the Lorentz symmetry is the Jüttner distribution.

It is known that the maximum entropy approach used in Ref [7] has the mathematical structure almost parallel to the traditional ensemble approach [17]. Therefore, the equilibrium distributions derived from ensemble approach can be examined with the same basis. This means the result in the present paper can be applicable to theories with the traditional approach.

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